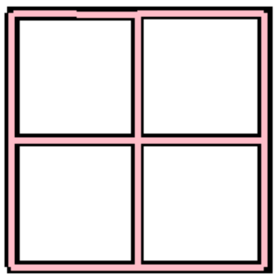
# SM4202 Exercise 2

1. Recall the *Gambler's ruin* example from the lectures. Suppose you play for the first four tosses only.
   (a) What is the probability that you win at least three times?
   (b) What are your expected winnings? *Hint: Count losses as negative winnings.*
   (c) Suppose you play for the first four tosses only, but in addition withdraw after your first loss. What are your expected winnings from this strategy?
   (d) Suppose you play for the first four tosses only, but in addition withdraw after your first win. What are your expected winnings from this strategy?

2. Consider a square $3 \times 3$ lattice of pipes. Suppose each of the 12 edges independently is open with probability $p$ and closed with probability $1 - p$. Compute the probability that there is no open path from the centre to the boundary.



3. (a) Consider a model which puts four points down in a unit disk, independently and uniformly at random. What is the probability that in the centred disk of radius 1/2 there are no points?
   (b) How does the answer change if now the model is that a Poisson (mean 1) number of points are put down in a unit disk, independently and uniformly at random?

4. Recall the *Brownian motion* example from lectures. Model the motion $X$ as follows: for $t \geq 0$, $s \geq 0$, we suppose $X_{t+s} - X_t$ has a normal distribution of mean zero and variance $s$, independent of behaviour of $X$ previous to time $t$.
   (a) If $X_0 = 1$ then what is the chance that $X_1$ is positive?
   (b) If $X_0 = 1$ then what is the chance that both $X_{1/2}$ and $X_1$ are positive?

5. Recall the *Changing words* example from lectures.
   (a) What is the state-space if the words have just one letter (and no emojis/'txting' are allowed!)?
   (b) Compute the probability of moving from one state to a different state in just one step, if the move is at all possible.
   (c) Exhibit a sequence of valid moves from `fail` to `pass`.

6. Consider two tireless and reasonably equally matched tennis players, $A$ and $B$. Sketch a diagrammatic description of the state-space for a random game played between $A$ and $B$.

7. Suppose that $X$ is a Markov chain with state-space $\{1, 2\}$. Prove the following simple consequence of the Chapman-Kolmogorov equations:

$$p_{ij}^{(3)} = p_{i1}p_{11}p_{1j} + p_{i1}p_{12}p_{2j} + p_{i2}p_{21}p_{1j} + p_{i2}p_{22}p_{2j}$$

8. (a) Give an example of a Markov chain which has just two communicating classes, and for which all but one of the states are inessential.
   (b) Construct an example of a Markov chain which has no essential states.
   (c) Construct an example of a Markov chain which has period 2 and yet for which no single state $i$ has $p_{ii}^{(2)} > 0$. Indicate what are its two cyclically moving sub-classes.

9. Consider the 4-state Markov chain, state-space $\{1, 2, 3, 4\}$, whose transition probability matrix is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 \\ 0 & 8/9 & 0 & 1/9 \end{pmatrix}$$

Divide the state-space into communicating classes and explain which classes are essential, which are transient, and which have periods (and what the periods are!).

10. In order to avoid runs of good luck for the player, a fruit machine owner alters his machine in the following way. If $W$ denotes 'win' for the player and $L$ denotes 'lose', then on any given play the probability of $W$ is arranged to be

| | | | | |
|---|---|---|---|---|
| 0.4 | if the last play was | $L$ | and the play before that | $L$ |
| 0.3 | ———————— | $L$ | ———————— | $W$ |
| 0.2 | ———————— | $W$ | ———————— | $L$ |
| 0.1 | ———————— | $W$ | ———————— | $W$ |

If $X_n$ denotes the outcome of the $n$-th play then the sequence of random vectors $Y = \{(X_n, X_{n+1}) \,|\, n \geq 1\}$ is a Markov chain. What is the transition probability matrix of Y ?

11. Consider the following simple model of weather. $X = \{X_0, X_1, \dots\}$ is a 2-state Markov chain such that $X_n = 1$ if it is wet on day $n$, $X_n = 0$ if it is dry on day $n$. Two of the transition probabilities are: P[wet tomorrow|wet today] $= 0.6$ and P[dry tomorrow|dry today] $= 0.5$.
    (a) Compute the other two transition probabilities.
    (b) Suppose that $X_0 = 0$ so initially it is dry. What is the probability that there follows a wet spell of exactly 4 days in length (so $X_0 = 0$, $X_1 = X_2 = X_3 = X_4 = 1$, $X_5 = 0$)?