SM4202 Exercise 2

1. Recall the *Gambler's ruin* example from the lectures. Suppose you play for the first four tosses only.

   Solution:

   Two gamblers A and B toss a sequence of fair coins. A gives B \$1 if tails, and the other way around if heads. Process stops whenever A or B has no more money.

   (a) What is the probability that you win at least three times?

   Solution:
   Let X be the number of wins. Each win is independent with probability $1/2$, so $X \sim \text{Bin}(4, 1/2)$. Then $\text{P}(X \geq 3) = \text{P}(X = 3, 4) = {}^4C_3(1/2)^4 + {}^4C_4(1/2)^4 = 5/16$.

   (b) What are your expected winnings? *Hint: Count losses as negative winnings.*

   Solution:
   We can argue by symmetry that the expected winnings is zero. More explicitly, the expected number of wins is $\text{E}(X) = np = \$2$. At the same time, let Y be the number of losses. Then $Y = 4 - X$. The expected number of losses is then $\text{E}(Y) = 4 - np = \$2$. Therefore the winnings is expected to be $\$2 - \$2 = \$0$.

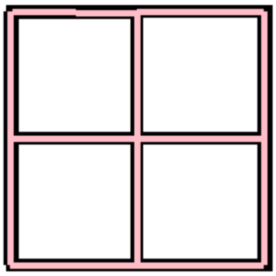
   (c) Suppose you play for the first four tosses only, but in addition withdraw after your first loss. What are your expected winnings from this strategy?

   Solution: Sum the results from the five possibilities (lose at first toss, lose at second, lose at third, lose at fourth, dont lose) each multiplied by the corresponding probability. The result is 0.

   (d) Suppose you play for the first four tosses only, but in addition withdraw after your first win. What are your expected winnings from this strategy?

   Solution: Same as (c)!

2. Consider a square 3×3 lattice of pipes. Suppose each of the 12 edges independently is open with probability p and closed with probability $1 - p$. Compute the probability that there is no open path from the centre to the boundary.



Solution: If this is to be true, then all four pipes leading from the centre must be closed. So the answer is $(1-p)^4$.

3. (a) Consider a model which puts four points down in a unit disk, independently and uniformly at random. What is the probability that in the centred disk of radius 1/2 there are no points?

Solution: The chance of any one point falling in the centred disk of radius 1/2 will be the ratio of areas of that disk to the unit disk, i.e. $\pi(1/2)^2/\pi = 1/4$. Consequently the total number of points (out of 4) falling in the centred disk of radius 1/2 is $\text{Bin}(4, 1/4)$. To have no points fall in the centred disk, the probability is $(3/4)^4$.

(b) How does the answer change if now the model is that a Poisson (mean 1) number of points are put down in a unit disk, independently and uniformly at random?

Solution: Now we are dealing with a Poisson number of points. Note that the actual number is unknown, but the mean is known to be 1. These points fall in the centred disk with probability 1/4 as before.

Suppose there are 0 points from the Poisson process, which occurs w.p. $e^{-1}1^0/0!$.

Suppose there is 1 point from the Poisson process, which occurs w.p. $e^{-1}1^1/1!$. This point does not fall in the centred disk w.p. 3/4.

Suppose there are 2 points from the Poisson process, which occurs w.p. $e^{-1}1^2/2!$. Both points do not fall in the centred disk w.p. $(3/4)^2$.

And so on...

The required probability is $\sum_{k=0}^{\infty} e^{-1}1^k/k! \times (3/4)^k = e^{-1}\sum_{k=0}^{\infty}(3/4)^k/k! = e^{-1}e^{3/4} = e^{-1/4}$.

4. Recall the *Brownian motion* example from lectures. Model the motion X as follows: for $t \geq 0$, $s \geq 0$, we suppose $X_{t+s} - X_t$ has a normal distribution of mean zero and variance s, independent of behaviour of X previous to time t.

(a) If $X_0 = 1$ then what is the chance that X_1 is positive?

Solution:
The problem boils down to figuring out $\mathrm{P}(X_1 > 0|X_0 = 1)$. We're not sure about the distribution of X_1 by itself, but the question tells us that $Y \equiv (X_1 - X_0) \sim \mathrm{N}(0, 1)$, thus

$$\begin{aligned}\mathrm{P}(X_1 > 0|X_0 = 1) &= \mathrm{P}(X_1 - X_0 > -X_0|X_0 = 1)\\ &= \mathrm{P}(Y > -1)\\ &= \Phi(1) = 0.8413\end{aligned}$$

where $\Phi(\cdot)$ is the CDF of a standard normal distribution.

(b) If $X_0 = 1$ then what is the chance that both $X_{1/2}$ and X_1 are positive?

Solution: We require $\mathrm{P}\left((X_{1/2} > 0) \cap (X_1 > 0)|X_0 = 1\right)$. Using conditional probabilities this is equal to $\mathrm{P}(X_1 > 0|X_0 = 1, X_{1/2} > 0)\,\mathrm{P}(X_{1/2} > 0|X_0 = 1)$. Since $V \equiv (X_{1/2} - X_0) \sim \mathrm{N}(0, 1/2)$, the second probability is equal to $\mathrm{P}(V > -1) = \int_{-1}^{\infty} e^{-v^2} \frac{\mathrm{d}v}{\sqrt{2\pi}}$.
Let $U \equiv (X_1 - X_{1/2}) \sim \mathrm{N}(0, 1/2)$. Then

$$\begin{aligned}\mathrm{P}(X_1 > 0|X_0 = 1, X_{1/2} > 0) &= \mathrm{P}(X_1 - (X_{1/2} - X_0) > -(X_{1/2} - X_0)|X_0 = 1, X_{1/2} > 0)\\ &= \mathrm{P}(U + 1 > -V)\\ &= \int_{-(v+1)}^{\infty} e^{-u^2} \frac{\mathrm{d}u}{\sqrt{2\pi}}\end{aligned}$$

Combining we get

$$\int_{v=-1}^{\infty} \int_{u=-(v+1)}^{\infty} e^{-(u^2+v^2)} \frac{\mathrm{d}u\,\mathrm{d}v}{2\pi}.$$

5. Recall the *Changing words* example from lectures.

(a) What is the state-space if the words have just one letter (and no emojis/'txting' are allowed!)?

Solution: What one letter words are there? These would be $\{a, i, o\}$. Perhaps 'e' for mathematicians?

(b) Compute the probability of moving from one state to a different state in just one step, if the move is at all possible.

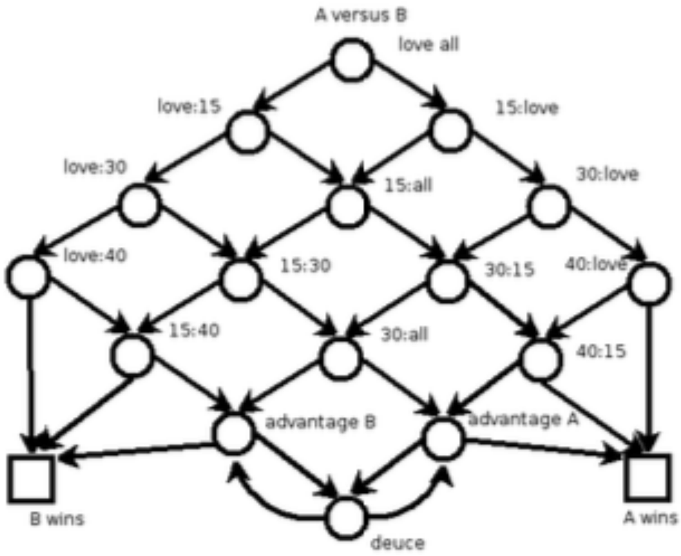
Solution: Assuming there are only 3 valid states, and there are 26 letters in the alphabet, the probability is 2/25.

(c) Exhibit a sequence of valid moves from `fail` to `pass`.

> **Solution:** `fail`, `pail`, `pall`, `pals`, `pass`.

6. Consider two tireless and reasonably equally matched tennis players, A and B. Sketch a diagrammatic description of the state-space for a random game played between A and B.

> **Solution:** The points in tennis are 'love' (i.e. 0), 15, 30, 40. When both players reach 30, then a deuce is reached, and the first player to gain two points ahead of their opponent wins. The figure indicates the state-space, with arrows connecting up the possible transitions. To be concise, I have amalgamated 30:40 and "advantage B", etc.



7. Suppose that X is a Markov chain with state-space $\{1, 2\}$. Prove the following simple consequence of the Chapman-Kolmogorov equations:

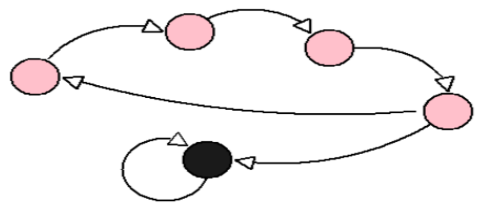
$$p_{ij}^{(3)} = p_{i1}p_{11}p_{1j} + p_{i1}p_{12}p_{2j} + p_{i2}p_{21}p_{1j} + p_{i2}p_{22}p_{2j}$$

> **Solution:**

$$
\begin{aligned}
p_{ij}^{(3)} = p_{ij}^{(2+1)} &= \sum_{k\in\{1,2\}} p_{ik}^{(2)} p_{kj}^{(1)} \\
&= p_{i1}^{(1+1)} p_{1j} + p_{i2}^{(1+1)} p_{2j} \\
&= \sum_{k\in\{1,2\}} p_{ik}^{(1)} p_{k1}^{(1)} p_{1j} + \sum_{k\in\{1,2\}} p_{ik}^{(2)} p_{k2}^{(1)} p_{2j} \\
&= p_{i1}p_{11}p_{1j} + p_{i2}p_{21}p_{1j} + p_{i1}p_{12}p_{2j} + p_{i2}p_{22}p_{2j}
\end{aligned}
$$

8. (a) Give an example of a Markov chain which has just two communicating classes, and for which all but one of the states are inessential.

Solution: Consider for example

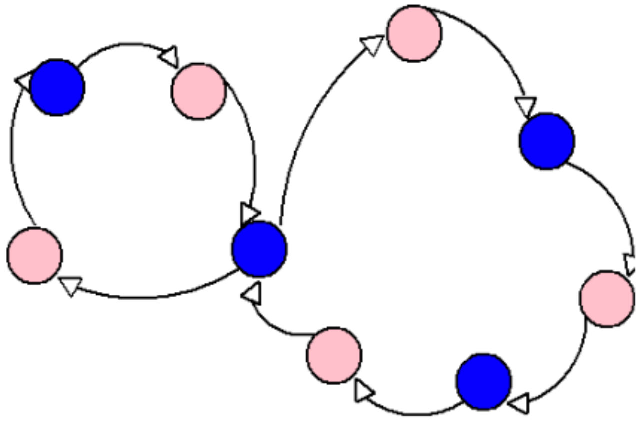


(b) Construct an example of a Markov chain which has no essential states.

Solution: Consider for example the rather trivial random walk in which it is possible to move to the left and impossible to move to the right:

(c) Construct an example of a Markov chain which has period 2 and yet for which no single state i has $p_{ii}^{(2)} > 0$. Indicate what are its two cyclically moving sub-classes.

Solution: Consider for example what happend if we join a loop chain of 4 states with a loop chain of 6 states.



The cyclically moving sub-classes are indicated by the different colours of states.

9. Consider the 4-state Markov chain, state-space $\{1, 2, 3, 4\}$, whose transition probability matrix is given by

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/4 & 1/4 \\ 0 & 1 & 0 & 0 \\ 0 & 8/9 & 0 & 1/9 \end{pmatrix}$$

Divide the state-space into communicating classes and explain which classes are essential, which are transient, and which have periods (and what the periods are!).

Solution:

(a) Communicating classes: $\{1\}, \{2, 3, 4\}$.

(b) Essential ("no arrows lead out"): $\{1\}$.

(c) Transient (inessential, some essential classes but not finite essential classes): $\{2, 3, 4\}$.

(d) Periods: $\{1\}$ has period 1, $\{2, 3, 4\}$ has period 1 (consider state 4: $p_{4,4} > 0$!).

10. In order to avoid runs of good luck for the player, a fruit machine owner alters his machine in the following way. If W denotes 'win' for the player and L denotes 'lose', then on any given play the probability of W is arranged to be

0.4	if the last play was	L	and the play before that	L
0.3	———	L	———	W
0.2	———	W	———	L
0.1	———	W	———	W

If X_n denotes the outcome of the n-th play then the sequence of random vectors $Y = \{(X_n, X_{n+1}) \,|\, n \geq 1\}$ is a Markov chain. What is the transition probability matrix of Y ?

Solution: The transition probability matrix is as follows, where the first position corresponds to (W, W), the second to (L, W), the third to (W, L), the fourth to (L, L). Here for example (L, W) means current play is W, previous is L.

$$\begin{array}{c} \\ WW \\ LW \\ WL \\ LL \end{array} \begin{array}{c} \begin{array}{cccc} WW & LW & WL & LL \end{array} \\ \begin{pmatrix} 0.1 & 0 & 0.9 & 0 \\ 0.2 & 0 & 0.8 & 0 \\ 0 & 0.3 & 0 & 0.7 \\ 0 & 0.4 & 0 & 0.6 \end{pmatrix} \end{array}$$

See this by arguing, for example, if $X_0 = (W, W)$ then the chance of W (leading to state (W, W)) is 0.1; the chance of L (leading to state (W, L)) is $0.9 = 10.1$; getting to either (L, W) or (L, L) is impossible, so has zero probability.

11. Consider the following simple model of weather. $X = \{X_0, X_1, \dots\}$ is a 2-state Markov chain such that $X_n = 1$ if it is wet on day n, $X_n = 0$ if it is dry on day n. Two of the transition probabilities are: P[wet tomorrow|wet today] = 0.6 and P[dry tomorrow|dry today] = 0.5.

(a) Compute the other two transition probabilities.

Solution: Use, for example, the fact that P[dry tomorrow|wet today] is equal to 1 − P[wet tomorrow|wet today] to deduce

$$\begin{aligned} \text{P[dry tomorrow|wet today]} &= 0.4 \\ \text{P[wet tomorrow|dry today]} &= 0.5 \end{aligned}$$

(b) Suppose that $X_0 = 0$ so initially it is dry. What is the probability that there follows a wet spell of exactly 4 days in length (so $X_0 = 0$, $X_1 = X_2 = X_3 = X_4 = 1$, $X_5 = 0$)?

Solution: The answer (using the Markov property) is

$$p_{0,1}(p_{1,1})^3 p_{1,0} = 0.5 \times 0.6^3 \times 0.4 = 0.0432.$$