SM4202 Exercise 3

1. Let X and Y be independent exponential random variables with *rates* λ and μ respectively. Find, for $x, y > 0$,
   (a) $\mathrm{P}(X < x)$;
   (b) $\mathrm{P}(X < Y)$;
   (c) $\mathrm{P}(X + Y > x + y | Y = y)$;
   (d) the distribution of $X \wedge Y =: \min(X, Y)$; and
   (e) $\mathrm{P}(X < Y | X \wedge Y = a)$.

2. Consider the following simplistic model of transitions between social classes as defined by Sociologists. Only males are considered, and by assumption every male has exactly one son. Let X_n denote the social class of the individual at generation n, and X_{n+1} the social class of his son, and so on. We assume that X_n forms a discrete-time Markov Chain with states $\{1, ..., s\}$ and one-step transition probabilities

$$\begin{aligned} p_{ij} &= \theta + (1-\theta)\phi_j \quad \text{for } i = j \\ p_{ij} &= (1-\theta)\phi_j \qquad \text{for } i \neq j \end{aligned}$$

   where $i, j = 1, \ldots, s$, $\phi_j > 0$ and $\sum_{j=1}^{2} \phi_j = 1$.

   Let state s denote the "highest" social class called 'toffs'. What is the expected number of generations taken by a family starting in social class 'toffs' to next be in this class? *Hint: You need only consider two states: 'toffs' and 'not toffs'.*

3. Which of the following possesses the Markov property? *Hint: is all you need to know contained in the information you have at time t?*
   (a) displacement $\{x_t \,|\, t \geq 0\}$ of a particle falling under constant non-zero gravitational attraction $\ddot{x} = -g$;
   (b) velocity $\{v_t t \,|\, t \geq 0\}$ of that same particle, $\dot{v} = -g$;
   (c) if a bus arrives at an exponentially distributed (rate 1) random time T, the random process X_t where $X_t = \mathbb{1}\{t \leq T\}$; and
   (d) as in (c) above, but T does not have an exponential distribution.

4. Consider the two-state Markov chain with Q-matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

   Write down the Kolmogorov equations. Verify by substitution that they are solved by

$$\begin{aligned} p_t(0,0) = p_t(1,1) &= e^{-t} \cosh t \\ p_t(0,1) = p_t(1,0) &= e^{-t} \sinh t. \end{aligned}$$

   Give a formula for $\mathrm{P}(X_{2\pi} = 1, X_\pi = 1 \,|\, X_0 = 1)$.

5. A Markov chain $X_t, t \geq 0$ with state space $\{0, 1\}$ has the following transition rates

$$q_{01} = 3, \qquad q_{10} = 5.$$

We assume that X satisfies the standing assumptions and that the regularity conditions of Kolmogorov forward differential equations hold.

(a) Write down its Q-matrix of rates.

(b) Write down both Kolmogorov backward and forward differential equations.

(c) Demonstrate that the transition probability functions $p_t(0, 0)$ and $p_t(1, 1)$ are given by

$$p_t(0, 0) = \frac{5 + 3e^{-8t}}{8}, \qquad p_t(1, 1) = \frac{3 + 5e^{-8t}}{8}.$$

(d) Compute the equilibrium (or long-run) distribution of X.