SM4202 Exercise 3

1. Let X and Y be independent exponential random variables with *rates* λ and μ respectively. Find, for $x, y > 0$,

   (a) $\mathrm{P}(X < x)$;

   Solution:

   $$\begin{aligned} \mathrm{P}(X < x) &= \int_0^x \lambda e^{-\lambda \tilde{x}} \,\mathrm{d}\tilde{x} \\ &= [-e^{-\lambda \tilde{x}}]_0^x \\ &= 1 - e^{-\lambda x}. \end{aligned}$$

   (b) $\mathrm{P}(X < Y)$;

   Solution: Recall the law of total probability $\mathrm{P}(A) = \sum_j \mathrm{P}(A|B_j)\,\mathrm{P}(B_j)$, where B_j is a partition of Ω. Take $A = (X < Y)$, and $B_j = (X = y)$. Then,

   $$\begin{aligned} \mathrm{P}(X < Y) &= \int_{y=0}^{\infty} \mathrm{P}(X < Y | X = y)\,\mathrm{P}(X \in \mathrm{d}y) \\ &= \int_{y=0}^{\infty} \mathrm{P}(Y > y)\,\mathrm{P}(X \in \mathrm{d}y) \\ &= \int_{y=0}^{\infty} e^{-\mu y} \lambda e^{-\lambda y} \,\mathrm{d}y \\ &= \lambda \int_0^{\infty} \frac{\lambda + \mu}{\lambda + \mu} e^{-(\lambda + \mu) y} \,\mathrm{d}y \\ &= \frac{\lambda}{\lambda + \mu}. \end{aligned}$$

   (c) $\mathrm{P}(X + Y > x + y | Y = y)$;

   Solution:

   $$\begin{aligned} \mathrm{P}(X + Y > x + y | Y = y) &= \mathrm{P}(X + y > x + y) \\ &= \mathrm{P}(X > x) \\ &= e^{-\lambda x}. \end{aligned}$$

   (d) the distribution of $X \wedge Y =: \min(X, Y)$; and

   Solution: Let $S = X \wedge Y$. Then $(S > x) \equiv (X > x) \cap (Y > x)$ (try drawing the two sets on an X-Y plane to see this).

$$\begin{aligned}
P(X \wedge Y) &= P\big((X > x) \cap (Y > x)\big) \\
&= P(X > x)\, P(Y > x) \quad \text{by independence} \\
&= e^{-\lambda x} e^{-\mu x} = e^{-(\lambda+\mu)x}.
\end{aligned}$$

Thus $X \wedge Y =: \min(X, Y) \sim \mathrm{Exp}(\lambda + \mu)$.

(e) $P(X < Y | X \wedge Y = a)$.

Solution:

$$\begin{aligned}
P(X < Y | X \wedge Y = a) &= \frac{P\big((X < Y) \cap (X \wedge Y = a)\big)}{P(X \wedge Y = a)} \\
&= \frac{P\big((X \in \mathrm{d}a) \cap (Y > a)\big)}{P(X \wedge Y = a)} \\
&= \frac{\lambda e^{-\lambda a} e^{-\mu a}}{(\lambda + \mu) e^{-(\lambda+\mu)x}} \\
&= \frac{\lambda}{\lambda + \mu}.
\end{aligned}$$

2. Consider the following simplistic model of transitions between social classes as defined by Sociologists. Only males are considered, and by assumption every male has exactly one son. Let X_n denote the social class of the individual at generation n, and X_{n+1} the social class of his son, and so on. We assume that X_n forms a discrete-time Markov Chain with states $\{1, ..., s\}$ and one-step transition probabilities

$$\begin{aligned}
p_{ij} &= \theta + (1 - \theta)\phi_j \quad \text{for } i = j \\
p_{ij} &= (1 - \theta)\phi_j \qquad \text{for } i \neq j
\end{aligned}$$

where $i, j = 1, \ldots, s$, $\phi_j > 0$ and $\sum_{j=1}^{s} \phi_j = 1$.

Let state s denote the "highest" social class called 'toffs'. What is the expected number of generations taken by a family starting in social class 'toffs' to next be in this class? *Hint: You need only consider two states: 'toffs' and 'not toffs'.*

Solution: Following the hint, we need only consider a reduced model with two states $\{1 = \text{Not toffs}, 2 = \text{Toffs}\}$. Then we have the probabilities

$$\begin{aligned}
p_{11} &= 1 - (1 - \theta)\phi & p_{12} &= (1 - \theta)\phi \\
p_{22} &= (1 - \theta)(1 - \phi) & p_{22} &= \theta + (1 - \theta)\phi
\end{aligned}$$

where $\phi := \phi_s$ and $(1 - \phi) = \sum_{j \neq s} \phi_j$.

Let X_n denote the state at the n'th generation, $N =$ No. of generations until first toff, and $E_i = \mathrm{E}(N|X_0 = i)$ for $i = 1, 2$. We are interested in E_2. We have

$$E_2 = 1 + (1-\theta)(1-\phi)E_1$$

Similarly,

$$\begin{aligned} E_1 &= 1 + \big(1-(1-\theta)\phi\big)E_1 \\ &= 1 + (1-\phi+\theta\phi)E_1 \\ \Rightarrow \not{1} &= 1/E_1 + \not{1} - \phi + \theta\phi \\ \Rightarrow E_1 &= 1/\phi(1-\theta). \end{aligned}$$

Thus

$$\begin{aligned} E_2 &= 1 + \frac{(1-\theta)(1-\phi)}{\phi(1-\theta)} \\ &= 1 + \frac{1-\phi}{\phi}. \end{aligned}$$

3. Which of the following possesses the Markov property? *Hint: is all you need to know contained in the information you have at time t?*
   (a) displacement $\{x_t \,|\, t \geq 0\}$ of a particle falling under constant non-zero gravitational attraction $\ddot{x} = -g$;
   (b) velocity $\{v_t \,|\, t \geq 0\}$ of that same particle, $\dot{v} = -g$;
   (c) if a bus arrives at an exponentially distributed (rate 1) random time T, the random process X_t where $X_t = \mathbb{1}\{t \leq T\}$; and
   (d) as in (c) above, but T does not have an exponential distribution.

Solution: (a) No. To calculate x you need $\dot{x}$.

(b) Yes. $v_t = v_0 - gt$.

(c) Yes. Exponential waiting times are memoryless.

(d) No. Only exponential waiting times are memoryless.

4. Consider the two-state Markov chain with Q-matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Write down the Kolmogorov equations. Verify by substitution that they are solved by

$$p_t(0,0) = p_t(1,1) = e^{-t}\cosh t$$
$$p_t(0,1) = p_t(1,0) = e^{-t}\sinh t.$$

Give a formula for $\mathrm{P}(X_{2\pi} = 1, X_\pi = 1 \,|\, X_0 = 1)$.

Solution: Note that $\sinh t = \frac{1-e^{-2t}}{2e^{-t}}$ while $\cosh t = \frac{1+e^{-2t}}{2e^{-t}}$.

Let $\mathbf{P}_t$ be a matrix with (i,j)-th entries equal to $\mathrm{P}(X_t = j|X_0 = i)$. The Kolmogorov forward and backward equations are given by $\dot{\mathbf{P}}_t = \mathbf{P}_t\mathbf{Q}$ and $\dot{\mathbf{P}}_t = \mathbf{Q}\mathbf{P}_t$ respectively.

Differentiating the above transition probabilities we obtain

$$\dot{p}_t(0,0) = \dot{p}_t(1,1) = e^{-t}\sinh t - e^{-t}\cosh t$$
$$\dot{p}_t(0,1) = \dot{p}_t(1,0) = e^{-t}\cosh t - e^{-t}\sinh t.$$

While the RHS of the (backward) equations are

$$\begin{pmatrix} e^{-t}\cosh t & e^{-t}\sinh t \\ e^{-t}\sinh t & e^{-t}\cosh t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} e^{-t}\sinh t - e^{-t}\cosh t & e^{-t}\cosh t - e^{-t}\sinh t \\ e^{-t}\cosh t - e^{-t}\sinh t & e^{-t}\sinh t - e^{-t}\cosh t \end{pmatrix}$$

Now

$$\begin{aligned} \mathrm{P}(X_{2\pi} = 1, X_\pi = 1 \,|\, X_0 = 1) &= \mathrm{P}(X_{2\pi} = 1 \,|\, \cancel{X_0 = 1}, X_\pi = 1)\,\mathrm{P}(X_\pi = 1 \,|\, X_0 = 1) \\ &= \mathrm{P}(X_\pi = 1 \,|\, X_0 = 1)^2 \\ &= p_\pi(1,1)^2 \\ &= (e^{-\pi}\sinh\pi - e^{-\pi}\cosh\pi)^2 \\ &= (-e^{-\pi}e^{-\pi})^2 = e^{-4\pi}. \end{aligned}$$

5. A Markov chain $X_t, t \geq 0$ with state space $\{0,1\}$ has the following transition rates

$$q_{01} = 3, \qquad q_{10} = 5.$$

We assume that X satisfies the standing assumptions and that the regularity conditions of Kolmogorov forward differential equations hold.

(a) Write down its Q-matrix of rates.

Solution:

$$Q = \begin{pmatrix} -3 & 3 \\ 5 & -5 \end{pmatrix}.$$

(b) Write down both Kolmogorov backward and forward differential equations.

Solution: They are $\dot{P}_t = QP_t$ and $\dot{P}_t = P_tQ$ respectively.

(c) Demonstrate that the transition probability functions $p_t(0,0)$ and $p_t(1,1)$ are given by

$$p_t(0,0) = \frac{5+3e^{-8t}}{8}, \qquad p_t(1,1) = \frac{3+5e^{-8t}}{8}.$$

Solution:
Step 1: Calculate eigenvalues of Q. This is done by solving $(-3-u)(-5-u)-15=0$, which yields $u=0,-8$.
Step 2: Find the eigenvectors of Q, one of which is $v_1 = (1,1)^\top$ associated with $u_1 = 0$. For the other one, solve

$$\begin{pmatrix} -3 & 3 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -8 \begin{pmatrix} a \\ b \end{pmatrix}$$

Giving us the equations

$$\begin{aligned} -3a + 3b &= -8a \\ 5a - 5b &= -8b \end{aligned}$$

which implies $-40a - 24b = 0 \Rightarrow a = -b(3/5)$. So $v_2 = (-3,5)^\top$ is an eigenvector.
Step 3: Solve V^{-1}:

$$\begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix}^{-1} = \frac{1}{8} \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}$$

Step 4: Solution is $P_t = e^{Qt}$.

$$\begin{aligned} P_t &= \frac{1}{8} \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{-8t} \end{pmatrix} \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} 5+3e^{-8t} & 3-3e^{-8t} \\ 5-5e^{-8t} & 3+5e^{-8t} \end{pmatrix} \end{aligned}$$

(d) Compute the equilibrium (or long-run) distribution of X.

Solution: As $t \to \infty$, $P_t \to A$, where

$$A = \frac{1}{8} \begin{pmatrix} 5 & 3 \\ 5 & 3 \end{pmatrix}$$